ELSEVIER

Topology and its Applications 106 (2000) 103–113

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

Lifting a generic surface in 3-space to an embedded surface in 4-space

Shin Satoh [1]

Department of Mathematics, Osaka City University, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585 Japan

Received 23 November 1998; received in revised form 28 December 1998

Abstract

We define a 'BW orientation' for a double point set of a generic surface in 3-space. Then we show that a given generic surface in 3-space is a projection of some embedded surface in 4-space if and only if its double point set admits a BW orientation. BW here means black/white; the BW orientation is motivated by checkerboard coloring of the complement of the surface.

Keywords: Double point set; Triple point; Branch point; Broken surface diagram; DP graph; BW orientation

AMS classification: 57Q45

1. Introduction

Throughout this paper, we fix a projection $\pi:\mathbb{R}^4\to\mathbb{R}^3$ by

$$\pi(x_1,x_2,x_3,t)=(x_1,x_2,x_3).$$

When we study an embedded surface in $\mathbb{R}^4$, we often use its projection onto $\mathbb{R}^3$ under π. Let $f:F\to\mathbb{R}^4$ be an embedding of a closed surface (compact without boundary) in $\mathbb{R}^4$. Then the map $g=\pi\circ f:F\to\mathbb{R}^3$ is taken to be *generic* by a slight perturbation of f; that is, the *double point set* of g,

$$\mathrm{cl}\{p\in\mathbb{R}^3\colon \sharp g^{-1}(p)>1\},$$

consists of isolated branch points, double point curves, and isolated triple points. The pre-image on F of the double point set is called a *double decker set* of g.

It is known that every generic surface in $\mathbb{R}^3$ is not a projection of some embedded surface in $\mathbb{R}^4$. A generic surface $g:F\to\mathbb{R}^3$ *lifts to an embedded surface in* $\mathbb{R}^4$ if there

[1] Email: susato@sci.osaka-cu.ac.jp.

0166-8641/00/$ – see front matter
PII: S0166-8641(99)00074-7

is an embedding $f : F \to \mathbb{R}^4$ such that $g = \pi \circ f$. We have several necessary and sufficient conditions for which generic surface lifts to an embedded surface:

(i) An existence of solutions of a 'system of linear inequations' induced from a double decker set (cf. [8,11]).
(ii) A 'colorability' for a double decker set (cf. [5,7]).
(iii) A 'compatible orientation' for a double point set (cf. [5,7]).

When we use the criteria (i) and (ii), we must know that which crossings in the double decker set are identified under the given generic map. As for the criterion (iii), it is applied for only orientable generic surfaces and, to be precise, it depends on not only a double point set but also its neighborhood in the generic surface.

In this paper, we discuss orientations of double point sets in relation to the lifting problem. Our criterion is similar to (iii), but it is essentially different from (iii) in that it is applied for every generic surfaces and in that it depends on double point sets only.

The paper is organized as follows. In Section 2 we review the notion of broken surface diagrams, signed triple points, signed and colored branch points, and so on. Section 3 is devoted to giving a new necessary and sufficient condition for a generic surface in $\mathbb{R}^3$ to be lifted to an embedded surface in $\mathbb{R}^4$. As an application, we give an elementary proof of Carter and Saito's formula between the numbers of triple points and branch points on a projection of an embedded surface in $\mathbb{R}^4$ (cf. [6]).

2. Preliminaries

In this section, we review the notion of embedded surfaces in $\mathbb{R}^4$ from the viewpoint of the diagrammatic theory. See [3–6] for more details.

Definition 2.1 (*Generic surfaces*). Let F denote a closed surface (compact without boundary). We say that a map $g : F \to \mathbb{R}^3$ is *generic* if for each point $p \in F$ there is a 3-ball neighborhood $N(p)$ containing p such that the pair $(N(p), g(F) \cap N(p))$ is homeomorphic to

- (B^3, the intersection of i coordinate planes) where B^3 is a 3-ball containing the origin ($i = 1, 2, 3$), or
- (B^3, the cone on a figure eight) where the figure eight curve is in the boundary of $N(p)$.

In the first case of Definition 2.1, the point $p \in F$ is called a *double point* if $i = 2$ or a *triple point* if $i = 3$. In the second case, the point p is called a *branch point*. See Fig. 1. Then the set $\mathrm{cl}\{y \in \mathbb{R}^3\colon\ \sharp g^{-1}(y) > 1\}$ consists of (possibly empty) isolated branch points, double point curves, and isolated triple points. This singular set is said to be a *double point set* and denoted by Γ_g.

Definition 2.2 (*Broken surface diagrams*). Let $f : F \to \mathbb{R}^4$ be an embedding such that the projection $\pi \circ f(F)$ is a generic surface. Consider a double point curve along which

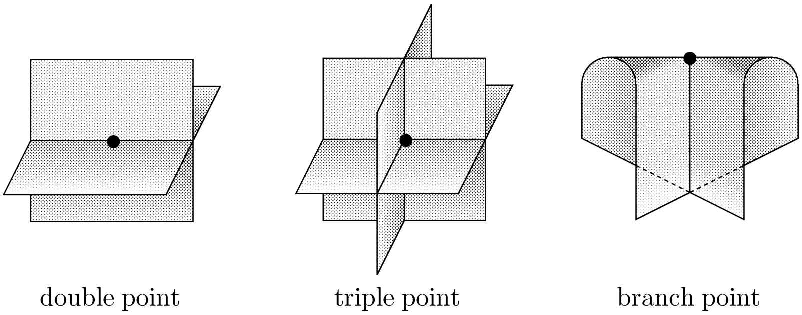


Fig. 1.

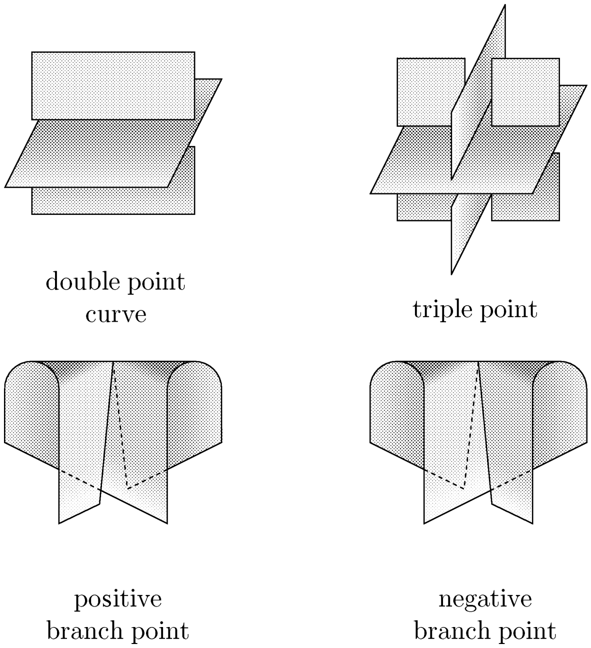


Fig. 2.

two sheets intersect. To indicate the crossing information—which of the two sheets along a double point curve is higher than the other with respect to the t-coordinate—we remove the neighborhood of the double point curve on the sheet (the *lower sheet*) which lies lower than the other sheet (the *upper sheet*). The obtained surface in $\mathbb{R}^3$ with no self-intersection is called a *broken surface diagram* associated with the embedded surface $f(F)$ in $\mathbb{R}^4$. See Fig. 2.

At an isolated triple point, such a breaking modification is as follows: the neighborhood of a triple point consists of three sheets. These sheets are labeled *top*, *middle* and *bottom*, and these indicate the relative position of the sheets with respect to the t-coordinate. In a broken surface diagram, the middle (respectively bottom) sheet is replaced with two (respectively four) disjoint disks which are obtained by cutting along the double point curves on the sheet. See Fig. 2 again.

Definition 2.3 (*Signs of branch points*). A *sign* of a branch point is defined when the branch point is on a generic projection of an embedded surface in $\mathbb{R}^4$. In the broken surface diagram associated with the embedding, there are two types of crossing information near a branch point—one is *positive* and the other is *negative*—depicted in Fig. 2.

When we assign $+1$ (respectively -1) to each positive (respectively negative) branch point, it is known that the sum of signs taken over all the branch points in the double point set is equal to the normal Euler number of the embedded surface in $\mathbb{R}^4$ (cf. [4]).

Definition 2.4 (*Lifts of generic surfaces*). A generic surface $g : F \to \mathbb{R}^3$ *lifts to an embedded surface* in $\mathbb{R}^4$ if there is an embedding $f : F \to \mathbb{R}^4$ such that $g = \pi \circ f$. Such an embedded surface in $\mathbb{R}^4$ is called a *lift* of the generic surface in $\mathbb{R}^3$.

Let a generic map $g : F \to \mathbb{R}^3$ be given. Assume that we can break the generic surface $g(F)$ along its double point set as depicted in Fig. 2. Then we easily construct an embedded surface in $\mathbb{R}^4$ that is a lift of $g(F)$—the construction of an embedded surface from a diagram is an inverse operation to constructing a diagram from an embedded surface. Hence we have the following.

Proposition 2.5 [5]. *A generic surface in $\mathbb{R}^3$ lifts to an embedded surface in $\mathbb{R}^4$ if and only if it admits a broken surface diagram.*

The criterion given in Proposition 2.5 is one theoretical answer to the lifting problem. Our criterion is based on this proposition.

Lemma 2.6 [1, Lemma 2.1]. *Let $g : F \to \mathbb{R}^3$ be a generic map. The generic surface $g(F)$ divides $\mathbb{R}^3$ into some regions. Then there exists a checker-board coloring* (*white and black*) *to the regions such that the adjacent regions receive the opposite colors.*

Proof. Let p_0 be a fixed point in $\mathbb{R}^3 \backslash g(F)$. For each region $D \subset \mathbb{R}^3 \backslash g(F)$, we take a point $p \in D$ and a regular curve $\gamma \subset \mathbb{R}^3 \backslash \Gamma_g$ connecting between p_0 and p such that γ intersects $g(F) \backslash \Gamma_g$ transversely in a finite number of points.

Let γ' be another curve as above. Since $H_1(\mathbb{R}^3) = 0$, the $\mathbb{Z}_2$-intersection number of the loop $\gamma \cup \gamma'$ and $g(F)$ in $\mathbb{R}^3$ is zero. Thus the parity of the number $\sharp(\gamma \cap g(F))$ does not depend on the chosen curve γ.

Now we can color D with black (respectively white) if $\sharp(\gamma \cap g(F))$ is even (respectively odd), then this coloring gives a required checker-board coloring. $\square$

In this paper, we fix such a coloring in Lemma 2.6. Signs of triple points and colors of branch points are defined using this checker-board coloring.

Definition 2.7 (*Signs of triple points*). A *sign* of a triple point is defined when the triple point is on a generic projection of an embedded surface in $\mathbb{R}^4$; pick a black region, B, among four of eight regions around a triple point. We choose $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ to be the normals to the top, middle, and bottom sheets respectively, and that they lie in B pointing into B. Then the triple point is defined to be *positive* if the orientation of $\mathbb{R}^3$ defined by the ordered triple $(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$ matches the right-handed orientation of $\mathbb{R}^3$, and *negative* otherwise. See Fig. 3.

Definition 2.8 (*Colors of branch points*). A *color* of a branch point is defined when the branch point is simply on a generic surface in $\mathbb{R}^3$ (it is not necessary that the generic surface is a projection of an embedded surface in $\mathbb{R}^4$). The neighborhood of a branch point on a generic surface in $\mathbb{R}^3$ looks like the cone on the figure eight. Then the branch point is *black* (respectively *white*) if the regions inside this figure eight is black (respectively white). See Fig. 4.

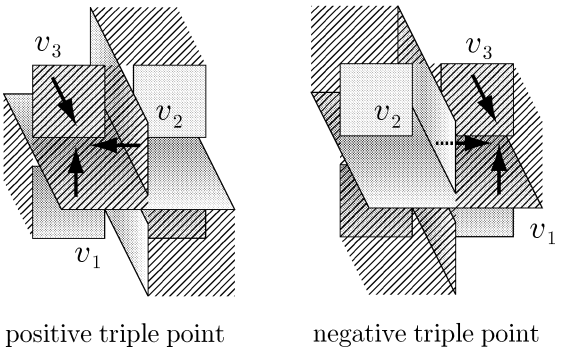


Fig. 3.

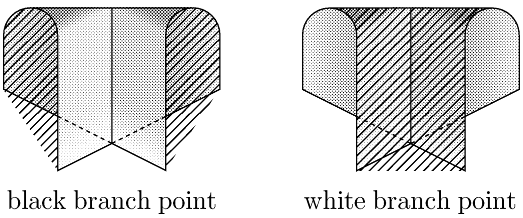


Fig. 4.

3. BW orientation

Before we state our main theorem, we give the notion of DP graphs and BW orientations.

Definition 3.1 (*DP graphs*). We consider a finite (possibly empty) graph with only 1- and 6-valent vertices. We set the six edges connecting to each 6-valent vertex in three pairs. A *DP graph* is a disjoint union of finite (possibly empty) circles and such a graph (including the designation of pairs of edges).

Notice that each circle of a DP graph is considered to consist of no vertex and one edge. Thus strictly speaking, a DP graph is *not* a graph in the graph theory.

An *embedded DP graph* in $\mathbb{R}^3$ is an embedding of the DP graph such that the image of the neighborhood of each 6-valent vertex looks like three arcs intersecting at one point and such that two edges in each of three designated pairs connect straight. Thus a model of a 6-valent vertex in a DP graph is the intersection of the coordinate axes. Designated pairs correspond to the axes.

Example 3.2. Let G be a bouquet with one vertex v and three edges e_1, e_2, e_3. We label the two end-parts of e_k the number k and k' $(k = 1, 2, 3)$. Then there are three types of DP graph $\Gamma = (G; S)$, where G is an underground graph and S is a designation of pairs of six arcs around v;

(1) $S = \{\{1, 2'\}, \{2, 3'\}, \{3, 1'\}\}$.
(2) $S = \{\{1, 1'\}, \{2, 3'\}, \{3, 2'\}\}$.
(3) $S = \{\{1, 1'\}, \{2, 2'\}, \{3, 3'\}\}$.

Then an embedded DP graph of Γ consists of one (case (1)), two (case (2)), three (case (3)) immersed circles. See Fig. 5.

The double point set of any generic surface in $\mathbb{R}^3$ is an embedded DP graph. Conversely Li shows the following.

Proposition 3.3 [9]. *For any embedded DP graph Γ in $\mathbb{R}^3$, there is a generic surface in $\mathbb{R}^3$ whose double point set is Γ.*

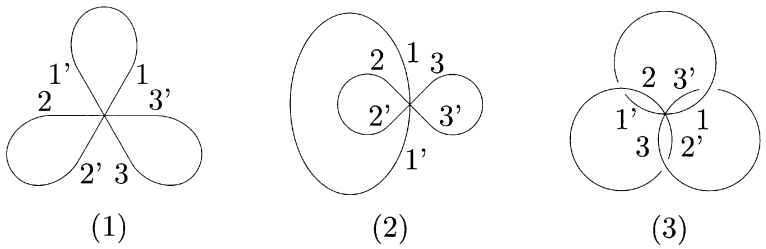


Fig. 5.

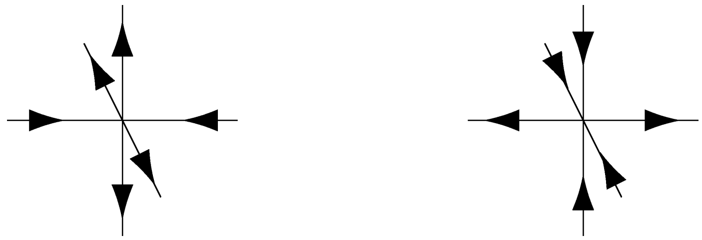


Fig. 6.

Definition 3.4 (*BW orientation*). A DP graph admits a *BW orientation* if every edge of the graph is oriented with the following conditions (i) and (ii) around every 6-valent vertex:

(i) Two edges in each of three designated pairs are oriented both into the 6-valent vertex or both out of the vertex.

(ii) The number of the edges whose orientations point into the 6-valent vertex is exactly two or exactly four.

A BW orientation of an embedded DP graph in $\mathbb{R}^3$ is induced from that of its original DP graph. See Fig. 6. By identifying a double point set of any generic surface in $\mathbb{R}^3$ with an embedded DP graph, we define a BW orientation which a double point set admits.

Now we are ready to state our main theorem.

Theorem 3.5. *A generic surface in $\mathbb{R}^3$ lifts to an embedded surface in $\mathbb{R}^4$ if and only if its double point set admits a BW orientation.*

Proof. Let $g : F \to \mathbb{R}^3$ denote a generic surface in $\mathbb{R}^3$. We use a checker-board coloring for the regions $\mathbb{R}^3 \backslash g(F)$.

($\Rightarrow$) Suppose that the generic surface has a lift that we fix for consideration. The four regions around each edge e in Γ_g are colored with black and white alternately. Pick a black region, B, among two of them. Let $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ be the normal vectors of two adjacent sheets to B pointing into B. These two sheets are disjointly embedded in $\mathbb{R}^4$ as the upper and lower sheets. We choose $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ to be the normals to the upper and lower sheets, respectively. We give the orientation $\boldsymbol{n}$ to e such that the right-handed orientation of $\mathbb{R}^3$ matches the triple $(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{n})$. See Fig. 7.

Then it is easy to check that this orientation satisfies the conditions of a BW orientation at a vertex ($=$ triple point) in Γ_g. More precisely, a positive (respectively negative) triple point has two (respectively four) edges pointing into the triple point. See Fig. 8. (Fig. 9 illustrates the BW orientation of edges which connect to branch points.) Hence the double point set Γ_g admits a BW orientation.

($\Leftarrow$) Suppose that the double point set Γ_g admits a BW orientation. For each oriented edge e, the vector $\boldsymbol{n}$ denote the orientation of e. We take two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ which are the normals of two sheets pointing into B, where B is the adjacent black region to e. We may assume that the right-handed orientation of $\mathbb{R}^3$ matches the triple $(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{n})$. Then we remove the neighborhood of e in the perpendicular sheet to $\boldsymbol{v}_2$. See Fig. 7 again.

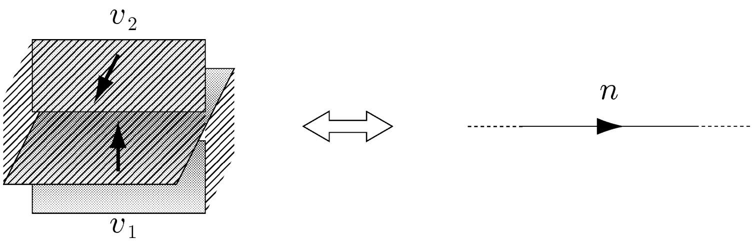
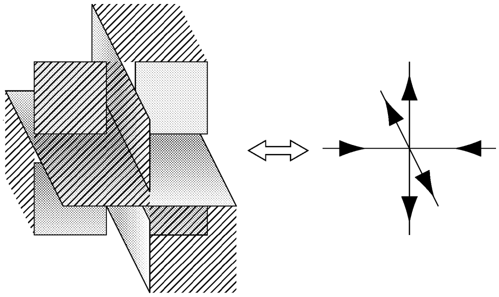
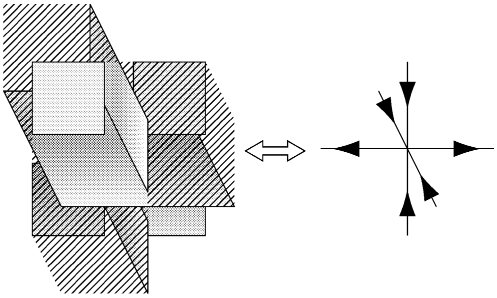


Fig. 7.



positive triple point



negative triple point

Fig. 8.

By the conditions of a BW orientation, this modification make a broken surface diagram of the generic surface $g(F)$. See Fig. 8 also again. This assures that the given generic surface lifts to an embedded surface in $\mathbb{R}^4$ by Proposition 2.5. $\square$

By Theorem 3.5, we immediately have the following.

Corollary 3.6. *Let Γ be a double point set of generic surface in $\mathbb{R}^3$ which has* (*respectively does not have*) *a lift. Then every generic surface in $\mathbb{R}^3$ with the double point set Γ also has* (*respectively does not have*) *a lift.*

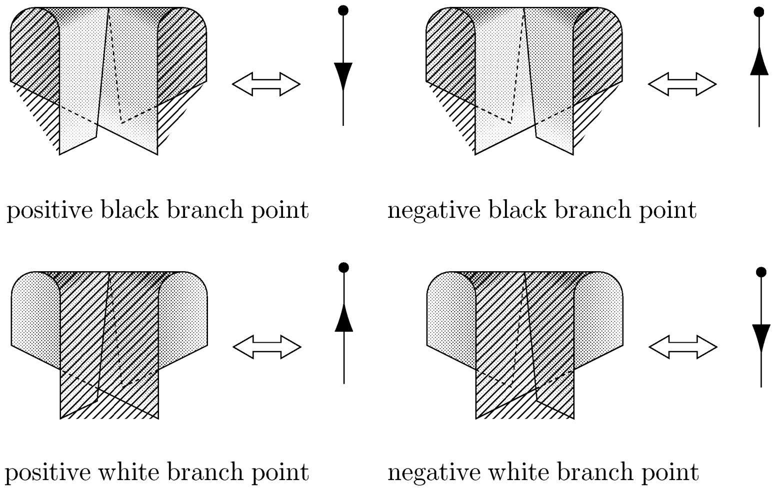


Fig. 9.

Corollary 3.7. *Let Γ be an embedded DP graph in $\mathbb{R}^3$. Suppose that Γ contains an immersed circle and that the number of pre-images of triple points on the circle is odd. Then every generic surface in $\mathbb{R}^3$ with the double point set Γ does not have any lift.*

Example 3.8. The double point set depicted in Fig. 5(1) consists of one immersed circle. Since the number of pre-images of the triple point on the circle is three, the double point set admits no BW orientation. Boy's surface has this double point set (cf. [2]), and so does not lift to an embedding in $\mathbb{R}^4$. More generally, it is shown that every immersed surface (that is, generic surface with no branch point) in $\mathbb{R}^3$ with a lift has even number of triple points.

Remark 3.9. Theorem 3.5 says that there is a one-to-one correspondence between BW orientations which the double point set of a generic surface admits and lifts of the generic surface. Two embedded surfaces in $\mathbb{R}^4$ are *equivalent* if there exists an orientation-preserving diffeomorphism of $\mathbb{R}^4$ which maps one to the other. Then different BW orientations of the double point set of a generic surface in $\mathbb{R}^3$ may give equivalent embedded surfaces in $\mathbb{R}^4$.

We observe Fig. 8 again. There are three sheets around a triple point on the generic projection of an embedded surface—a top sheet, a middle sheet, and a bottom sheet. Each sheet contains four edges of a double point set. On the top sheet (or the bottom sheet), the two of four edges point into the vertex. In the middle sheet, the four edges point all into the vertex or point all out of the vertex. Hence we have the following. (In [10] we use the special case of Proposition 3.10.) See also Fig. 10.

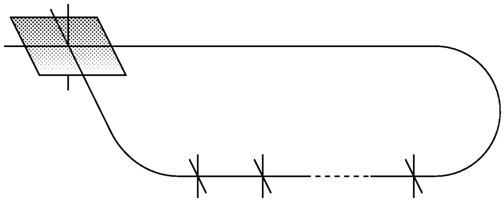


Fig. 10.

Proposition 3.10. *Let $g : F \to \mathbb{R}^3$ denote a generic surface in $\mathbb{R}^3$. Let t be a triple point in Γ_g and H one of the three sheets around t. Suppose that Γ_g contains an immersed arc $\alpha : [0,1] \to \Gamma_g$ such that*

(i) *$\alpha(0) = \alpha(1) = t$,*

(ii) *the short end-parts $\beta_0 = \alpha([0,\varepsilon])$ and $\beta_1 = \alpha([1-\varepsilon, 1])$ are on H for a small $\varepsilon > 0$, and*

(iii) *β_0 and β_1 are perpendicular to each other at t.*

If the number of the pre-images of triple points on α is even (*respectively odd*)*, then H is necessarily a top sheet or a bottom sheet* (*respectively a middle sheet*) *for any lift of $g(F)$.*

Notations 3.11 [6]. For a generic projection of an embedded surface in $\mathbb{R}^4$,

- T_+ (respectively T_-) = the number of the positive (respectively negative) triple points.
- B_+ (respectively B_-) = the number of the positive (respectively negative) black branch points.
- W_+ (respectively W_-) = the number of the positive (respectively negative) white branch points.
- $T = T_+ - T_-$, $B = B_+ - B_-$ and $W = W_+ - W_-$.

As we notice in Section 2, the sum $B + W$ is equal to the normal Euler number of the embedded surface in $\mathbb{R}^4$ (cf. [4]). On the other hand, Carter and Saito show the following. We give an elementary proof of this formula.

Theorem 3.12 [6]. *For an embedded surface in $\mathbb{R}^4$, we have*

$$T = (W - B)/2.$$

Proof. Let Γ^* be a double point set of the projected generic surface in $\mathbb{R}^3$ except for the simple closed curves. Consider a BW orientation of Γ^* induced from the given embedded surface in $\mathbb{R}^4$. We count the number of the edges in Γ^* by two ways (see Figs. 8 and 9): the number of the edges oriented into the vertices is

$$2T_+ + 4T_- + B_- + W_+,$$

and the number of the edges oriented out of the vertices is

$$4T_+ + 2T_- + B_+ + W_-.$$

These numbers coincide and so we have

$$2(T_+ - T_-) = (W_+ - W_-) - (B_+ - B_-). \qquad \square$$

References

[1] J.J. Nuño Ballesteros, O. Saeki, On the number of singularities of a generic surfaces with boundary in a 3-manifold, Hokkaido Math. J., to appear.

[2] W. Boy, Über die curvature integra und die topologie geschlossener flaschen, Math. Ann. 57 (1903) 151–184.

[3] J.S. Carter, How Surfaces Intersect in Space: An Introduction to Topology, 2nd edn., World Scientific Publishing, Singapore, 1995.

[4] J.S. Carter, M. Saito, Canceling branch points on projections of surfaces in 4-space, Proc. Amer. Math. Soc. 116 (1) (1992) 229–237.

[5] J.S. Carter, M. Saito, Knotted Surfaces and Their Diagrams, Mathematical Surveys and Monographs, Vol. 55, Amer. Math. Soc.

[6] J.S. Carter, M. Saito, Normal Euler classes of knotted surfaces and triple points on projections, Proc. Amer. Math. Soc. 125 (2) (1997) 617–623.

[7] J.S. Carter, M. Saito, Surfaces in 3-space that do not lift to embeddings in 4-space, Knot Theory, Banach Center Publications, Vol. 42, Institute of Mathematics, Polish Academy of Sciences, Warszawa, 1998, pp. 29–47.

[8] C. Giller, Towards a classical knot theory for surfaces in $\mathbb{R}^4$, Illinois J. Math. 26 (4) (1982) 591–631.

[9] Li, Gui-Song, On the self-intersections of immersed surfaces in $\mathbb{R}^3$, Proc. Amer. Math. Soc., to appear.

[10] S. Satoh, On non-orientable surfaces in 4-space which are projected with at most one triple point, Proc. Amer. Math. Soc., to appear.

[11] Yamagata, Some conditions and some examples for maps which are projections of 2-knots, Master thesis, Hiroshima Univ., 1996 (in Japanese).